

Spin squeezing, entanglement and coherence in two driven, dissipative, nonlinear cavities coupled with single and two-photon exchange

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We investigate spin squeezing, quantum entanglement and second order coherence in two coupled, driven, dissipative, nonlinear cavities. We compare these quantum statistical properties for the cavities coupled with either single or two-photon exchange. Solving the quantum optical master equation of the system numerically in the steady state, we calculate the zero-time delay second-order correlation function for the coherence, genuine two-mode entanglement parameter, and an optimal spin squeezing inequality associated with particle entanglement. We identify regimes of distinct quantum statistical character depending on the relative strength of photon-exchange and nonlinearity. Moreover, we examine the effects of weak and strong drives on these quantum statistical regimes.

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I. INTRODUCTION

Multiphoton processes in quantum optical systems have been intensely studied [1, 2] due to their central role in modern applications such as quantum switching [3], quantum communication and computation [4]. They are also used for fundamental explorations of phase transitions in coupled nonlinear cavity or superconducting (SC) circuit quantum electrodynamics (QED) systems [5–19].

An earlier study of two nonlinear cavities coupled with single-photon exchange [20] revealed a curious interplay between coherence and localization of the photons. It is concluded that photons are coherent and delocalized over the cavities when the tunneling exchange is stronger than the nonlinearity. In the opposite case of weaker tunneling, photons are localized in each cavity and anti-bunched [21].

In the present work, we address the question if such an interplay can go across to quantum correlations as well. We specifically ask how such mutual influences between localization, coherence and quantum entanglement [22] change under two-photon exchange [23–25] as well as under strong drive conditions [26, 27]. Realization of two-photon exchange coupling was proposed for circuit QED systems [28–33].

Profound relations between different definitions of spin squeezing and entanglement have already been explored in detail [34–38]. Using Lipkin-Meshkov-Glick (LMG) Model [39], interplay of quantum correlations in squeezing and entanglement in finite quantum systems was discussed quite recently [35]. Entanglement dynamics and exact properties of LMG model in the thermodynamical limit have been carefully analyzed in Refs. [40–42]. Two photon exchange coupled cavities can also be described by the LMG model [24]. Our present contribution estab-

lishes further connections between coherence and mode entanglement to spin squeezing and pairwise entanglement in this model under drive and dissipation.

We calculate and compare the zero time delay second order quantum coherence function [43], an optimal spin squeezing inequality associated with pairwise entanglement [44] and the genuine mode entanglement parameter [45]. Mode entanglement occurs in the second quantization description of the system; and hence it is fundamentally different from the particle entanglement, which happens in the first quantization description [46–48]. Some possible realizations and experiments to detect multiparticle entanglement via optimal spin squeezing inequalities can be found in Refs. [49–51].

This paper is organized as follows. In Sec. II we describe the single and two photon exchange coupled nonlinear cavity models under consideration. In Sec. III we introduce the parameters that we calculate to characterize coherence, entanglement and spin squeezing and present their steady state results. We conclude in Sec. IV.

II. THE MODEL SYSTEM

We consider a system of two identical nonlinear cavities, labeled with $i = 1, 2$, coupled either by single or two photon exchange interactions. Both cavities are driven by a coherent pump at rate F at the laser frequency ω_L . The corresponding model Hamiltonians in a frame rotating at ω_L can be written as

$$\begin{aligned} \hat{H}^{(1)} = & \sum_{i=1,2} (U \hat{b}_i^\dagger \hat{b}_i^\dagger \hat{b}_i \hat{b}_i + F \hat{b}_i^\dagger + F^* \hat{b}_i) \\ & + J(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_1 \hat{b}_2^\dagger), \end{aligned} \quad (1)$$

$$\begin{aligned} \hat{H}^{(2)} = & \sum_{i=1,2} (U \hat{b}_i^\dagger \hat{b}_i^\dagger \hat{b}_i \hat{b}_i + F \hat{b}_i^\dagger + F^* \hat{b}_i) \\ & + J(\hat{b}_1^{2\dagger} \hat{b}_2^2 + \hat{b}_2^{2\dagger} \hat{b}_1^2), \end{aligned} \quad (2)$$

where U is the nonlinearity parameter and J is the photon exchange coefficient. Here we denote the model with

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single and two photon exchange coupling as $\hat{H}^{(1)}$ and $\hat{H}^{(2)}$, respectively. These models describe generic two-site Kerr-Hubbard type interactions that may be realized in settings other than coupled cavities, for example in ultracold atoms [52–56]. Annihilation (creation) operator for the cavity photons with frequency ω_i is denoted by \hat{b}_i (\hat{b}_i^\dagger). We take $\omega_i = \omega_L$.

Let us re-express the model Hamiltonians using the pseudo-spin operators of the cavity fields,

$$\begin{aligned}\hat{J}_x &\equiv \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1), \\ \hat{J}_y &\equiv \frac{-i}{2}(\hat{b}_1^\dagger \hat{b}_2 - \hat{b}_2^\dagger \hat{b}_1), \\ \hat{J}_z &\equiv \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2).\end{aligned}\quad (3)$$

They satisfy the SU(2) spin algebra $[\hat{J}_\alpha, \hat{J}_\beta] = \epsilon^{\alpha\beta\gamma} \hat{J}_\gamma$. Here $\alpha, \beta, \gamma \in \{x, y, z\}$ and $\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita density. We obtain

$$\hat{H}^{(1)} = U\hat{J}_z^2 + 2J\hat{J}_x + F\hat{b}_i^\dagger + F^*\hat{b}_i, \quad (4)$$

$$\hat{H}^{(2)} = U\hat{J}_z^2 + 2J(\hat{J}_x^2 - \hat{J}_y^2) + F\hat{b}_i^\dagger + F^*\hat{b}_i. \quad (5)$$

While the tunneling term is a mere rotation operator in the case of single photon exchange, it is a generator of spin squeezing that redistributes the spin fluctuations by twisting them about the two axis [57] in the case of two photon exchange. The nonlinear term always acts as a generator of spin squeezing by twisting the fluctuations around a single (z) axis. Spin squeezing is associated with multi-particle entanglement. Another type of entanglement can be found between the cavity modes. This mode entanglement is enforced by the mode coupling character of the tunneling terms. Drive term brings coherence into the system.

In the next section we define the measures of quantum correlations to calculate spin squeezing, entanglement and coherence properties of these models. All the measures are closely related to the spin noise and hence strong interplay between coherence, squeezing and types of entanglement is expected.

III. RESULTS AND DISCUSSIONS

To investigate the quantum dynamics of our model systems under dissipation, we assume that the coupling of the cavity photons and the reservoir photons is weak; and the correlation time of the reservoir photons is negligibly short. Under these so called Born and Markov conditions, the dynamics of the open systems can be determined by solving the master equations

$$\dot{\hat{\rho}}^{(j)} = -i[\hat{H}^{(j)}, \hat{\rho}^{(j)}] + \sum_{i=1,2} \kappa_i \mathcal{D}[x] \hat{\rho}^{(j)}, \quad (6)$$

for both single ($j = 1$) and two photon ($j = 2$) exchange cases. Here, $\hat{\rho}^{(j)}$ is the density operator for the corresponding case; κ_i are the photon loss rates out of the

cavities. $\mathcal{D}[x]\hat{\rho}^{(j)} = [2\hat{x}\hat{\rho}^{(j)}\hat{x}' - \hat{x}'\hat{x}\hat{\rho}^{(j)} - \hat{\rho}^{(j)}\hat{x}'\hat{x}]/2$ are the Liouvillian superoperators in the Lindblad form. We assume $\kappa_i \equiv \kappa$ and coherent pump amplitude F is taken to be real.

We solve the master equation for the steady state density operator using the QuTiP: Quantum Toolbox in Python software [58]. The master equation is solved by taking $N_1 = N_2 = 4$ for the Fock space dimensions of each cavity field. When we increase the Fock space dimension up to 8 and remake our calculations, we find that the results remain unchanged. We consider two cases distinguished by the action of weak $F/\kappa = 0.1$ MHz and strong pump $F/\kappa = 1$ MHz. We take $\kappa/2\pi = 0.4$ MHz. The parameters we use in the simulations are within the ranges accessible in present circuit QED systems [59, 60].

Numerically computed steady state density operator is used for calculation of second order coherence, spin squeezing and mode entanglement parameters. Zero-time delay second order quantum coherence function is defined by [43]

$$g^{(2)}(0) = \frac{\text{tr}(\hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b})}{[\text{tr}(\hat{b}^\dagger \hat{b})]^2}, \quad (7)$$

$$= 1 + \frac{\langle (\Delta \hat{n})^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle^2}, \quad (8)$$

Here $\hat{n} = \hat{b}^\dagger \hat{b}$, $\Delta \hat{n} = \hat{n} - \langle \hat{n} \rangle$; and we use ρ instead of $\rho^{(j)}$ for notational simplicity. We evaluate $g^{(2)}(0)$ for $\hat{b} = \hat{b}_i$. Both cavity photons have identical coherence functions due to exchange symmetry of our models under $b_1 \leftrightarrow b_2$. $g^{(2)}(0) < 1$ implies antibunching and sub-Poissonian statistics of photons, while $g^{(2)}(0) > 1$ implies bunching and super-Poissonian statistics. Poissonian statistics and coherent photons are recognized by $g^{(2)}(0) = 1$. Antibunching condition is a violation of the Cauchy-Schwartz inequality obeyed by classical light; and hence antibunching is a profound manifestation of quantum light.

Spin squeezing is witnessed by the inequality [44]

$$\langle \hat{J}_k^2 \rangle + \langle \hat{J}_l^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta \hat{J}_m)^2, \quad (9)$$

where k, l, m take all the possible permutations of x, y, z . Inequality given in Eq. (9) is one of the four optimal spin squeezing inequalities introduced in Ref. [44] with N being the total number of particles in the system. Violation of it implies spin squeezing and particle entanglement. We rewrite the optimal spin squeezing inequality given in Eq. (9) with $k = x, l = z$ and $m = y$ as

$$\zeta \equiv \langle \hat{J}_x^2 \rangle + \langle \hat{J}_z^2 \rangle - \frac{N}{2} - (N-1)(\Delta \hat{J}_y)^2 \leq 0. \quad (10)$$

The positive values of ζ indicate spin squeezing and pairwise entanglement in the first quantization description. As we have an open system, $N = \langle \hat{N} \rangle = \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle$, with $\hat{N} = \hat{n}_1 + \hat{n}_2$ and $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$, is not conserved and varies in time.

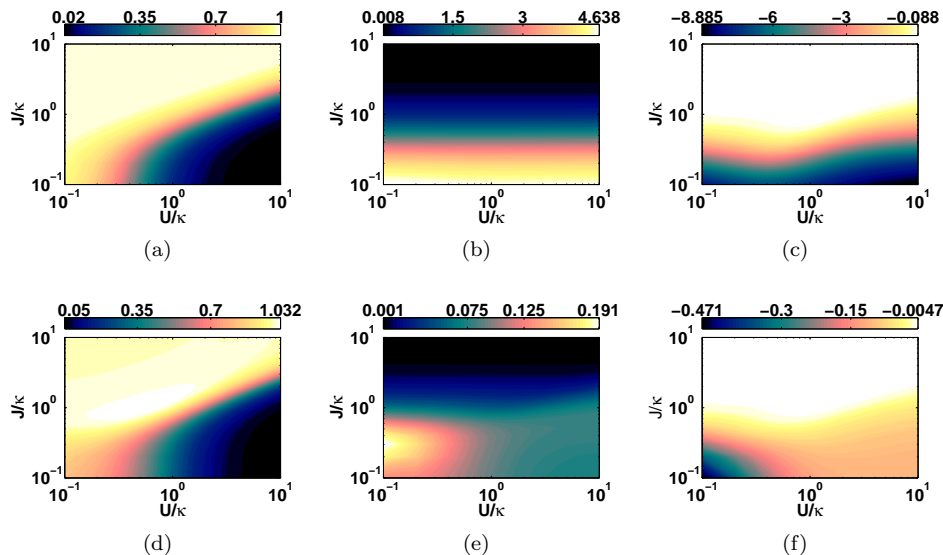


FIG. 1: (Color Online) Dependence of (a) $g^{(2)}(0)$, (b) ζ (c) λ_{12} for $F/\kappa = 0.1$ and (d) $g^{(2)}(0)$, (e) ζ (f) λ_{12} for $F/\kappa = 1$ with respect to dimensionless J/κ and U/κ in two-site KH system with single-photon exchange. ζ is multiplied by 10^3 and λ_{12} is multiplied by 10^5 for visibility.

Pairwise entanglement correspondence of spin squeezing can be difficult to interpret for photons due to lack of a commonly accepted definition of photon wave function in position representation, especially in free space. Here we simply assume that the localized cavity fields can be considered as photonic wave functions. As the photons are identical particles, their pairwise entanglement may be ambiguously determined by the spin squeezing inequality for some cases and moreover it may not be a useful resource for quantum metrology applications [48, 61]. In the spirit of Ref. [62], however, this should not prevent us to calculate spin squeezing for the purpose of distinguishing pairwise entanglement from the mode entanglement and to explore their relation to coherence.

We investigate the entanglement in the second quantized description in terms of the genuine two-mode entanglement parameter λ_{12} defined by [45]

$$\lambda_{12} = |\langle \hat{b}_1^\dagger \hat{b}_2 \rangle|^2 - \langle \hat{n}_1 \hat{n}_2 \rangle, \quad (11)$$

where $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$ with $i = 1, 2$. Positivity of the parameter λ_{12} indicates mode entanglement.

An intuitive link between coherence, spin squeezing and mode entanglement can be formed by considering that coherence function is a relative measure of number fluctuations which carries information on deviation from Poisson statistics. One can establish an analogy between the number operator \hat{n} and the z-component of the angular momentum operator \hat{J}_z using $\hat{n}_1 = \hat{N}/2 + \hat{J}_z$. The coherence function is influenced by squeezing the spin noise around the z axis. Under stringent conditions spin squeezing implies particle entanglement. Particle like behavior, enforced by the photon antibunching can be as-

sociated with spin squeezing type pairwise nonlinear interactions. In that case coherence and spin squeezing interplay can be further extended to multiparticle entanglement. Similarly λ_{12} can be expressed in terms of spin fluctuations and thus would be influenced by the coherence and spin squeezing in the system.

Above heuristic arguments motivates the existence of an interplay between coherence, spin squeezing, pairwise and mode entanglement. Let us now analyze specific cases numerically. We will first consider the single photon exchange case, then proceed to two photon exchange case in the following subsections.

A. Single-Photon Exchange

In Fig. 1(a), Fig. 1(b) and Fig. 1(c) we plot second order coherence, spin squeezing and mode entanglement parameters as functions of nonlinearity and photon exchange coefficients in the case of weak drive with $F/2\pi = 0.04$ MHz, respectively. Fig. 1(a) reproduces the result in Ref. [20]. Second order coherence varies over the range of $0.00183 \leq g^{(2)}(0) \leq 1.011$. Coherent delocalization of cavity photons happens for $J/\kappa > 1$ and $J > U$, which corresponds to the photon bunching and Poissonian statistics indicated by $g^{(2)}(0) \sim 1$. Strong photon antibunching regime with $g^{(2)}(0) \sim 0$ lies in the region of $U/\kappa > 1$ and $J/\kappa < 1$.

Fig. 1(b) shows that spin squeezing always present in the system for the ranges of $0.1 < U/\kappa, J/\kappa < 10$. Pairwise nonlinearity measured by U in the driven system is strong enough for the survival of pairwise entanglement

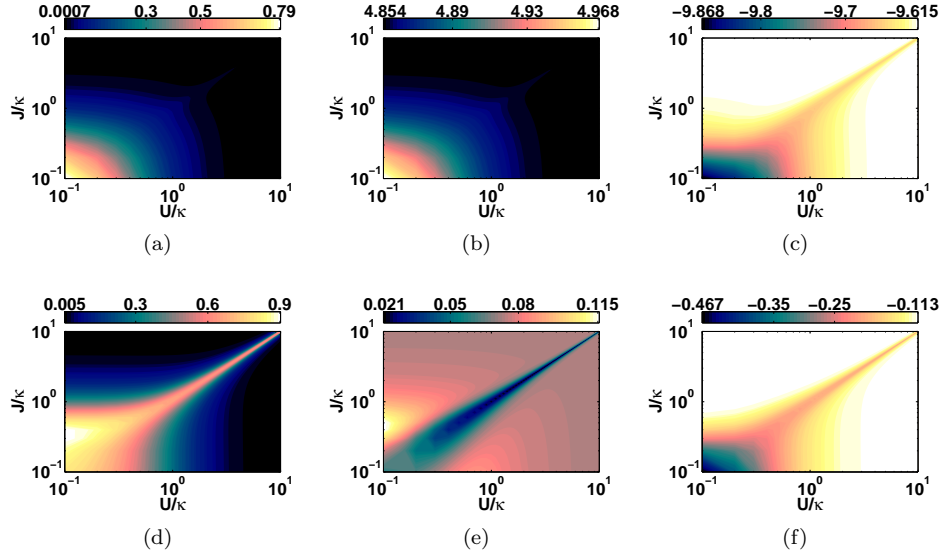


FIG. 2: (Color Online) Dependence of (a) $g^{(2)}(0)$, (b) ζ (c) λ_{12} for $F/\kappa = 0.1$ and (d) $g^{(2)}(0)$, (e) ζ (f) λ_{12} for $F/\kappa = 1$ with respect to dimensionless J/κ and U/κ in two-site KH system with two-photon exchange. ζ is multiplied by 10^3 and λ_{12} is multiplied by 10^5 for visibility.

in the steady state. It is easier to violate spin squeezing inequality in the region of $J/\kappa < 1$, where antibunching and hence particle like behavior is more significant. This is intuitively expected as wave like character enforced by the delocalizing photon hopping makes spin squeezing and particle entanglement more difficult, while nonlinear interaction favors pairwise particle correlations.

Mode entanglement parameter λ_{12} behavior with J and U , shown in Fig. 1(c), is opposite to that of spin squeezing. Lowest and highest values of λ_{12} emerge in the antibunching and bunching regions of Fig. 1(a), respectively. Localization of photons in the cavities favor pairwise particle entanglement; while wave like behavior associated with coherent delocalization by the photon exchange favors mode correlations. Though λ_{12} approaches toward positive values as J/κ increase; mode correlations remained in the steady state are not strong enough for entanglement for the range of J/κ we considered. In comparison to Fig. 1(b), we see that particle correlations are more robust against cavity loss.

Fig. 1(d), Fig. 1(e) and Fig. 1(f) depict quantum coherence, spin squeezing and mode entanglement for the case of strong drive of $F/2\pi = 0.4$ MHz, respectively. Coherent effects of the drive is harmful for the particle like character and the range of $g^{(2)}(0)$ is shifted away from 0 and becomes $0.00486 \leq g^{(2)}(0) \leq 1.0321$. Stronger nonlinearity is needed for antibunching and thus the region of antibunching is narrowed in the U/κ axis relative to that in Fig. 1(a). Spin squeezing and pairwise entanglement becomes harder to establish relative to weak drive case. The maximum value of ζ reduces about an order of magnitude when the coherent drive is increased an or-

der of magnitude. Spin squeezing is almost absent in the bunching region. On the other hand, mode entanglement is approximately established in the same bunching region.

It should be stressed that the single photon exchange model gives clearly distinct roles to the interactions. Nonlinearity is localizing and a single-axis twisting spin squeezing type of interaction. Photon hopping is delocalizing and a mode correlating interaction. Two photon hopping however has particle and delocalizing character. It is delocalizing due to photon exchange effect, but also establishes pairwise correlations by a two axis twisting interaction. We will point out emergence of distinct relation between second order coherence and entanglement in the following subsection.

B. Two-Photon Exchange

In Fig. 2, we show our results for two-photon exchange interaction. Fig. 2(a), Fig. 2(b) and Fig. 2(c) depict quantum coherence, spin squeezing and mode entanglement for $F/2\pi = 0.04$ MHz, respectively. In contrast to single photon exchange, there is no coherent bunching region in Fig. 2(a) and $g^{(2)}(0)$ varies in the range of $0.00723 \leq g^{(2)}(0) \leq 0.7944$. The antibunching is strong in both delocalizing photon hopping and local cavity nonlinearity dominated regimes. Relatively weak antibunching occurs in $J/\kappa, U/\kappa < 1$ region where coherent drive is comparable to these local and nonlocal pairwise interactions.

In the two photon exchange case, analogous to LMG

model, there are both single axis and two axis twisting routes to spin squeezing. When $J/\kappa > 1$ the former becomes the major route leading to spin squeezing; while when $U/\kappa > 1$ the latter becomes the main interaction causing spin squeezing. Accordingly strong violation of spin squeezing occurs over the entire domain of $J/\kappa, U/\kappa$. Similar level of violation can only be found in the narrow strip of weak J/κ in Fig. 1(b). The uniformity of the pairwise entanglement witness parameter with U which is present in Fig. 1(b) is no longer present in Fig. 2(b). Variation of spin squeezing and antibunching strength with hopping and nonlinearity becomes identical for the case of two photon exchange.

Complementary character of mode and pairwise entanglement is again revealed in comparison of Fig. 2(c) and Fig. 2(b). Regions of minimum violation of spin squeezing inequality coincides with the regions where mode entanglement parameter is maximum. Mode correlations are much weaker in the case of two photon exchange as can be seen by comparing the maximum of ζ in Fig. 2(c) to that of Fig. 1(c). Variation of $g^{(2)}(0)$ with J/κ and U/κ in comparison to that of λ_{12} turns out to be of reciprocal nature for two photon exchange case. The largest mode entanglement parameter regions are the same with strongest antibunching regions. This reflects the particle like character of delocalization induced by two photon hopping.

Fig. 2(d), Fig. 2(e) and Fig. 2(f) show quantum coherence, spin squeezing and mode entanglement for $F/2\pi = 0.4$ MHz, respectively. Strong drive helps to bring coherence into the system and now the maximum $g^{(2)}(0) \sim 1$. The separation of two strong antibunching regions by a weak antibunching region around the line $J = U$ becomes more visible. Similar lines divides the spin squeezing and mode entanglement figures into regions with same statistical character as well. Strong drive works against spin squeezing and pairwise entanglement; while it helps to establish mode correlations. The correlations are still not entangled in Fig. 2(f); and pairwise entanglement, though becomes harder to establish, survives in Fig. 2(e).

IV. CONCLUSION

We considered single photon and two photon exchange coupled cavities with Kerr nonlinearity under drive and dissipation. We examined the steady state second order coherence, spin squeezing and pairwise entanglement as well as mode entanglement.

Complementary character of pairwise and mode entanglement is revealed. Drive strength can be used to tune between mode and particle correlations. Relations between localization and bunching as well as delocalization and antibunching is found to be translated to pairwise and mode entanglement in the case of single photon exchange. Due to its nonlocal but pairwise interaction nature, such direct relations becomes more subtle in the case of two photon exchange, where delocalization has particle character. Equivalent behavior with nonlinearity and photon hopping is found between coherence and spin squeezing. Pairwise entanglement and strong antibunching ranges are enhanced towards the strong photon exchange regime with the two photon coupling. In addition to identification of coherent and correlated photons, second order coherence can discriminate the type of these correlations.

Our results illuminates the subtle relations between locality, coherence and quantum correlations that could be exploited for synthesis of many body quantum entangled states in coupled cavities with designed short and long range interactions.

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